

Geometrical approach to qunit decoherence

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Abstract

In the following paper we generalize the geometrical framework of a qubit decoherence model to higher dimensions. A quantum mixed state is represented by a probability distribution which is the Kähler function on the projective Hilbert space. The Markovian master equation turns out to be the Fokker-Planck equation for the quantum probability distribution. Several examples are analyzed.

1 Introduction

The geometrical formulation of quantum mechanics [1, 2, 3, 4] recognizes the projective Hilbert space $\mathbb{P}\mathcal{H}$ as the space of quantum states. For every point in $\mathbb{P}\mathcal{H}$ there exists the corresponding rank-1 projector $|\psi\rangle\langle\psi|$ and a ray in the Hilbert space \mathcal{H} passing through ψ (see also [5, 6, 7, 8]). If we choose $\mathcal{H} = \mathbb{C}^n$, then the space of states

$$\mathbb{P}\mathcal{H} = \mathbb{C}P^{n-1} = U(n)/U(n-1) \quad (1)$$

is an $n-1$ -dimensional complex space, equipped with the Fubini-Study metric g and the symplectic form ω , such that

$$\mathcal{K} = g + i\omega. \quad (2)$$

The triple $(\mathbb{C}P^{n-1}, g, \omega)$ is the Kähler space [9]. On this space one can define the Kähler functions [10] in the following way. A function $f : \mathbb{C}P^{n-1} \rightarrow \mathbb{C}$ is Kählerian iff its Hamiltonian vector field X_f , which is given by the equation $df = \omega(X_f, \cdot)$, is a Killing vector field, that is, iff it satisfies $\mathcal{L}_{X_f}g = 0$. These functions form a linear subspace in the space of all functions $\mathcal{F}(\mathbb{C}P^{n-1}) := \{f : \mathbb{C}P^{n-1} \rightarrow \mathbb{C}\}$.

With every operator $A \in \mathcal{B}(\mathcal{H})$ one can associate the function $f_A : \mathbb{C}P^{n-1} \rightarrow \mathbb{C}$ given by

$$f_A([\psi]) := \frac{\langle\psi|A|\psi\rangle}{\langle\psi|\psi\rangle}. \quad (3)$$

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Note that, if we take the operators $A, B \in \mathcal{B}(\mathcal{H})$, $C = -2i[A, B]$, and $D = 2[A, B]_+ := 2(AB + BA)$, then the corresponding Kähler functions f_A, f_B, f_C, f_D are connected to each other by

$$f_C = \{f_A, f_B\} = -X_{f_A}(f_B), \quad f_D = \{f_A, f_B\}_+, \quad (4)$$

where $\{f_A, f_B\} := \omega(X_{f_A}, X_{f_B})$ is the Poisson bracket on $\mathbb{C}P^{n-1}$ defined by the symplectic form ω , and $\{f_A, f_B\}_+ := g(X_{f_A}, X_{f_B}) + 4f_A f_B$ is the symmetric bracket [4] given by the Fubini-Study metric g .

Let us introduce $(\psi_0, \psi_1, \dots, \psi_{n-1}) \in \mathbb{C}^n$ in terms of the octant coordinate system,

$$(\psi_0, \psi_1, \dots, \psi_{n-1}) = (N_0, N_1 e^{i\nu_1}, \dots, N_{n-1} e^{i\nu_{n-1}}), \quad (5)$$

where $0 \leq \nu_i \leq 2\pi$ and $\sum_{i=0}^{n-1} N_i = 1$. In local coordinates we set

$$\begin{cases} N_0 = \cos \theta_1 \sin \theta_2 \sin \theta_3 \dots \sin \theta_{n-1}, \\ N_1 = \sin \theta_1 \sin \theta_2 \sin \theta_3 \dots \sin \theta_{n-1}, \\ N_2 = \cos \theta_2 \sin \theta_3 \dots \sin \theta_{n-1}, \\ \vdots \\ N_{n-1} = \cos \theta_{n-1} \end{cases} \quad (6)$$

with $0 \leq \theta_i \leq \pi/2$. Now, the Fubini-Study metric g and the symplectic form ω read

$$\begin{aligned} g &= dN_0^2 + \sum_{i=1}^{n-1} \left[dN_i^2 + N_i^2(1 - N_i^2) d\nu_i^2 - 2 \sum_{j=i+1}^{n-1} N_i^2 N_j^2 d\nu_i d\nu_j \right], \\ \omega &= \sum_{i=1}^{n-1} N_i dN_i \wedge d\nu_i. \end{aligned} \quad (7)$$

The symplectic form defines a volume element on $\mathbb{C}P^{n-1}$,

$$\text{Vol}(\mathbb{C}P^{n-1}) = \int_{\mathbb{C}P^{n-1}} \omega = \frac{\pi^{n-1}}{(n-1)!}. \quad (8)$$

Observe that, from the geometrical point of view, N_k 's form the positive hyperoctant of an $(n-1)$ -sphere, whereas the phases ν_k 's form an $(n-1)$ -torus [6].

Now, for a given density operator ρ , let us introduce the following function,

$$p([\psi]) := \frac{(n-1)!}{\pi^{n-1}} \frac{\langle \psi | \rho | \psi \rangle}{\langle \psi | \psi \rangle}, \quad (9)$$

so that $p([\psi])$ is a probability distribution on $\mathbb{C}P^{n-1}$, which means that $p([\psi]) \geq 0$ and

$$\int_{\mathbb{C}P^{n-1}} p([\psi]) \omega = 1, \quad (10)$$

where ω is given by (7). This function corresponds to a legitimate density operator iff p is a Kähler function. It is worth noting that p describes a pure state $\rho = |\psi\rangle\langle\psi|$ iff $p([\psi]) = (n-1)!/\pi^{n-1}$.

Our goal is to apply the introduced geometric structures to the problem of Markovian evolution and decoherence processes of open quantum systems. Section 2 provides a short description of general qudit decoherence, whose special cases are then discussed in Sections 3-5. Final conclusions are gathered in the last section.

2 Time-evolution of Open Quantum Systems

The Markovian master equation has the well-known GKSL form [11, 12],

$$\dot{\rho}_t = -i[H, \rho_t] + \frac{1}{2} \sum_{k=0}^{n^2-1} \gamma_k \left(V_k \rho_t V_k^\dagger - \frac{1}{2} [V_k^\dagger V_k, \rho_t]_+ \right), \quad (11)$$

where H – the effective Hamiltonian of the quantum system, V_k – the time-dependent noise operators satisfying $\text{tr } V_k^\dagger V_l = \delta_{kl}$, $\text{tr } V_k = n\delta_{0k}$, γ_k – the time-dependent decoherence rates. We will be interested only in the non-Hamiltonian evolution, therefore limiting our discussion to

$$\dot{\rho}_t = L_t \rho_t = \frac{1}{2} \sum_{k=0}^{n^2-1} \gamma_k \left(V_k \rho_t V_k^\dagger - \frac{1}{2} [V_k^\dagger V_k, \rho_t]_+ \right), \quad (12)$$

where L is the generator of the evolution. The above equation can be rewritten with the use of double bracket structures,

$$\dot{\rho}_t = -\frac{1}{8} \sum_{k=1}^{n^2-1} \gamma_k \left([V_k^\dagger, [V_k, \rho_t]] + [V_k, [V_k^\dagger, \rho_t]] + [V_k^\dagger, [V_k, \rho_t]_+] - [V_k, [V_k^\dagger, \rho_t]_+] \right), \quad (13)$$

which, in the geometrical language, is equivalent to the following equation for the probability distribution,

$$\dot{p}_t = \frac{1}{16} \sum_{k=1}^{n^2-1} \gamma_k \left(|X_{v_k}|^2 p_t + \text{Im } X_{v_k} \{v_k^*, p_t\}_+ \right). \quad (14)$$

Here, $p_t = \langle \rho_t \rangle$, $v_k = \langle V_k \rangle$, and v_k^* denotes the complex conjugation of v_k . The above formula is clearly the Fokker-Planck equation [13] for the probability distributions of quantum states.

The time-evolution $\rho_0 \mapsto \rho_t$ can be given by the following completely positive trace-preserving (CPT) map Λ_t ,

$$\rho_t = \Lambda_t[\rho_0] = \sum_{k=0}^{n^2-1} \pi_k A_k \rho_0 A_k^\dagger, \quad (15)$$

with A_k - the Kraus operators, and π_k - the probability distribution, i.e. $\sum_{k=0}^{n^2-1} \pi_k(t) = 1$, $\pi_0(0) = 1$. It is possible to rewrite equation 15 in the form of

$$\rho_t - \rho_0 = \sum_{k=0}^{n^2-1} \gamma_k \left(A_k \rho_t A_k^\dagger - \frac{1}{2} [A_k^\dagger A_k, \rho_t]_+ \right), \quad (16)$$

which, in the geometrical language, corresponds to

$$p_t - p_0 = \frac{1}{8} \sum_{k=1}^{n^2-1} \pi_k \left(|X_{a_k}|^2 p_0 + \text{Im } X_{a_k} \{a_k^*, p_0\}_+ \right) \quad (17)$$

with $p_t = \langle \rho_t \rangle$ and $a_k = \langle A_k \rangle$.

In the precious work [14], the authors analyzed the properties of the random unitary qubit evolution. Now, we would like to generalize this picture to higher dimensions. However, it turns out that the noise operators in random unitary evolution lose Hermiticity for $n > 2$. Therefore, we need to consider three generalizations: the one that keeps unitarity (*Weyl operators*), Hermiticity (*Gell-Mann matrices*), or both (*tensor products of Pauli matrices*).

3 Weyl operators

When we are interested in generalizing the two-dimensional case in a way that the unitarity of the noise operators is preserved, a natural choice is the following form of the master equation,

$$\dot{\rho}_t = \frac{1}{2} \sum_{k_1, k_2=0}^{n-1} \gamma(U_{k_1 k_2}) \left(U_{k_1 k_2} \rho_t U_{k_1 k_2}^\dagger - \rho_t \right). \quad (18)$$

Here, $\gamma(U_{k_1 k_2})$ is just another symbol for $\gamma_{k_1 k_2}$. The noise operators $U_{k_1 k_2}$ are the Weyl operators, which are defined by (c.f. [16])

$$U_{k_1 k_2} := \sum_{m=0}^{n-k_2-1} \Omega^{k_1 m} E_{m, m+k_2} + \sum_{m=n-k_2}^{n-1} \Omega^{k_1 m} E_{m, m+k_2-n}, \quad (19)$$

with the coefficient $\Omega = \exp\left(\frac{2\pi i}{n}\right)$, $k_1, k_2 = 0, \dots, n-1$, and E_{ml} being the matrix with a 1 in the (m, l) th entry and 0 elsewhere.

The solution of (18) induces the dynamical equation for p_t ,

$$\dot{p}_t = \sum_{k_1, k_2=0}^{n-1} \left(\frac{\gamma(u_{k_1 k_2}) + \gamma(u_{k_1 k_2}^*)}{32} |X_{u_{k_1 k_2}}|^2 p_t + \frac{\gamma(u_{k_1 k_2}) - \gamma(u_{k_1 k_2}^*)}{32} \text{Im } X_{u_{k_1 k_2}} \{u_{k_1 k_2}^*, p_t\}_+ \right). \quad (20)$$

In the special case, where $\gamma(u_{k_1 k_2}) = \gamma(u_{k_1 k_2}^*)$, the second component under the sum symbol vanishes, and therefore we have

$$\dot{p}_t = \frac{1}{16} \sum_{k_1, k_2=0}^{n-1} \gamma(u_{k_1 k_2}) |X_{u_{k_1 k_2}}|^2 p_t. \quad (21)$$

The eigenvalues $l_{k_1 k_2}$ of (21) to the eigenfunctions $u_{k_1 k_2}$ read

$$l_{k_1 k_2} = \frac{1}{2} \sum_{j_1, j_2=0}^{n-1} \gamma(u_{k_1 k_2}) \left(\operatorname{Re} \Omega^{k_2 j_1 - k_1 j_2} - 1 \right). \quad (22)$$

From the above equation, we get the following conditions for the positivity of the solution,

$$\sum_{j_1, j_2=0}^{n-1} \gamma(u_{k_1 k_2}) \left(1 - \operatorname{Re} \Omega^{k_2 j_1 - k_1 j_2} \right) \geq 0. \quad (23)$$

The CPT map that describes the evolution $\rho \rightarrow \rho_t$ reads as follows,

$$\Lambda_t(\rho) = \sum_{k_1, k_2=0}^n \pi(U_{k_1 k_2}) U_{k_1 k_2} \rho U_{k_1 k_2}^\dagger. \quad (24)$$

It describes the Markovian evolution iff it satisfies the BLP conditions [15],

$$\frac{d}{dt} \|\Lambda_t(\rho_1 - \rho_2)\|_{\text{tr}} \leq 0, \quad (25)$$

where ρ_1 and ρ_2 are arbitrary initial states. Note that (23) agrees with the BLP conditions calculated in [16].

4 Gell-Mann matrices

By choosing the generalization in which the Hermiticity of the noise operators is preserved, the formula (12) can be rewritten with the use of a double commutator structure,

$$\dot{\rho}_t = -\frac{1}{4} \sum_{k_1, k_2=0}^{n-1} \gamma(\tau_{k_1 k_2}) [\tau_{k_1 k_2}, [\tau_{k_1 k_2}, \rho_t]], \quad (26)$$

where $\tau_{k_1 k_2}$ are the Gell-Mann matrices defined as follows [17],

$$\begin{aligned} \tau_{k_1 k_2}^S &= E_{k_1 k_2} + E_{k_2 k_1}, & 0 \leq k_1 < k_2 \leq n-1, \\ \tau_{k_1 k_2}^A &= -i(E_{k_1 k_2} - E_{k_2 k_1}), & 0 \leq k_1 < k_2 \leq n-1, \\ \tau_{k_1 k_1}^D &= \sqrt{\frac{2}{k_1(k_1+1)}} \left(\sum_{j=0}^{k_1-1} E_{jj} - k_1 E_{k_1 k_1} \right), & 1 \leq k_1 \leq n-1, \\ \tau_{00}^D &= \sum_{j=0}^{n-1} E_{jj}. \end{aligned} \quad (27)$$

In the geometrical formulation, the master equation (26) takes the form of

$$\dot{p}_t = \frac{1}{16} \sum_{k_1, k_2=0}^{n-1} \gamma(f_{k_1 k_2}) \{f_{k_1 k_2}, \{f_{k_1 k_2}, p_t\}\} = \frac{1}{16} \sum_{k_1, k_2=0}^{n-1} \gamma(f_{k_1 k_2}) X_{f_{k_1 k_2}}^2 p_t = \frac{1}{4} \Delta_\gamma p_t, \quad (28)$$

where $f_{k_1 k_2} = \langle \tau_{k_1 k_2} \rangle$ are the observable functions corresponding to the Gell-Mann matrices, and $\Delta_\gamma := \frac{1}{4} \sum_{k_1, k_2=0}^{n-1} \gamma(f_{k_1 k_2}) X_{f_{k_1 k_2}}^2$ is the generalized Laplacian.

In general, the eigenfunctions of the generator are time-dependent. This makes finding the eigenvalue formulas highly problematic. However, if we narrow down our interest to the case in which $\gamma_{k_1 k_2}^S + \gamma_{k_1 k_2}^A =: \gamma$, then it turns out that $f_{k_1 k_2}$'s are the desired eigenfunctions. Their corresponding eigenvalues are listed below:

$$\begin{aligned} l_{k_1 k_2}^{S/A} &= -\gamma_{k_1 k_2}^{A/S} - \frac{k_1}{2(k_1 + 1)} \gamma_{k_1 k_1}^D - \sum_{j=k_1+1}^{k_2-1} \frac{\gamma_{jj}^D}{2j(j+1)} - \frac{k_2 + 1}{2k_2} \gamma_{k_2 k_2}^D - \frac{\gamma}{2}(n-2), \quad 0 \leq k_1 < k_2 \leq n-1, \\ l_{k_1 k_1}^D &= -\frac{\gamma}{2}n, \quad 1 \leq k_1 \leq n-1, \\ l_{00}^D &= 0. \end{aligned} \quad (29)$$

In the isotropic case, when $\gamma_{k_1 k_2} =: \gamma$, equation (26) simplifies to

$$\dot{p}_t = \frac{\gamma}{16} \sum_{k_1, k_2=0}^{n-1} X_{f_{k_1 k_2}}^2 p_t = \frac{\gamma}{4} \Delta p_t, \quad (30)$$

with Δ being the Laplacian on $\mathbb{C}P^{n-1}$. This corresponds to the dephasing channel for a qunit. Moreover, every isotropic evolution whose noise operators are given by a unitary rotation of the Gell-Mann matrices leads to the same dynamical equation for p_t .

Lemma 1. *Every dipole Kähler function $f_{k_1 k_2}$, i.e. $k_1 \neq 0$ and $k_2 \neq 0$, is an eigenfunction of the Laplace operator,*

$$\Delta f_{k_1 k_2} = -4n f_{k_1 k_2}. \quad (31)$$

Proof. Let us start from the isotropic ($\gamma_k = 1$) algebraical master equation 26 for a traceless density operator ρ ,

$$\dot{\rho} = -\frac{1}{4} \sum_{k=1}^{n^2-1} [\tau_k, [\tau_k, \rho]], \quad (32)$$

where τ_k are the Gell-Mann matrices, and

$$\rho = \sum_{k=1}^{n^2-1} x_k \tau_k. \quad (33)$$

Using the properties of τ_k (see e.g. [6]),

$$[\tau_i, \tau_j] = 2i \sum_{k=0}^{n^2-1} \epsilon_{ijk} \tau_k, \quad (34)$$

$$\sum_{i,j=0}^{n^2-1} \epsilon_{ijk} \epsilon_{ijl} = n \delta_{kl}, \quad (35)$$

we arrive at

$$\dot{\rho} = -\frac{1}{4} \sum_{k,l=1}^{n^2-1} x_l [\tau_k, [\tau_k, \tau_l]] = - \sum_{i,j,k,l=1}^{n^2-1} x_l \tau_i \epsilon_{jkl} \epsilon_{jki} = -n\rho. \quad (36)$$

In the geometrical framework, the above equation is equivalent to

$$\dot{p} = \frac{1}{4} \Delta \rho = -n\rho. \quad (37)$$

Therefore, the eigenvalue equation for the Laplace operator reads

$$\Delta f_k = -4n f_k \quad (38)$$

for $k \neq 0$. □

5 Tensor products of Pauli matrices

It turns out that there is a way to keep the noise operators in higher dimensions both unitary and Hermitian. To make that possible, however, we need to limit our discussion to very specific dimensions of the Hilbert space. Let us write down the following master equation,

$$\dot{\rho}_t = \frac{1}{2} \sum_{k_1, \dots, k_N=0}^3 \gamma(\eta_{k_1 \dots k_N}) \left(\eta_{k_1 \dots k_N} \rho_t \eta_{k_1 \dots k_N} - \rho_t \right), \quad (39)$$

where the noise operators $\eta_{k_1 \dots k_N}$ are tensor products of Pauli matrices σ_{k_j} ,

$$\eta_{k_1 \dots k_N} = \bigotimes_{j=1}^N \sigma_{k_j}, \quad (40)$$

and the Hilbert space dimension $n = 2^N$. Similarly to the case of the Gell-Mann matrices, the master equation (39) reduces to a simpler formula with the double commutator structure,

$$\dot{\rho}_t = -\frac{1}{4} \sum_{k_1, \dots, k_N=0}^3 \gamma(\eta_{k_1 \dots k_N}) [\eta_{k_1 \dots k_N}, [\eta_{k_1 \dots k_N}, \rho_t]]. \quad (41)$$

In the geometrical formulation, the master equation (39) takes the form of

$$\dot{p}_t = \frac{1}{16} \sum_{k_1, \dots, k_N=0}^3 \gamma(m_{k_1 \dots k_N}) X_{m_{k_1 \dots k_N}}^2 p_t = \frac{1}{4} \Delta_\gamma p_t, \quad (42)$$

where $m_{k_1 \dots k_N} = \langle \eta_{k_1 \dots k_N} \rangle$, and $\Delta_\gamma := \frac{1}{4} \sum_{k_1, \dots, k_N=0}^3 \gamma(m_{k_1 \dots k_N}) X_{m_{k_1 \dots k_N}}^2$.

The CPT map satisfying (39) has the following Kraus representation,

$$\rho_t = \Lambda_t[\rho] = \sum_{k_1, \dots, k_N=0}^3 \pi(\eta_{k_1 \dots k_N}) \eta_{k_1 \dots k_N} \rho_0 \eta_{k_1 \dots k_N}, \quad (43)$$

which is equivalent to the following equation for p_t ,

$$p_t = \left(\frac{1}{2} \Delta_\pi + 1 \right) p_0. \quad (44)$$

Substituting (44) into (42), we obtain the following formula describing the dependence between γ 's and π 's,

$$\left[\Delta_{\dot{\pi}} - \frac{1}{2} \Delta_\gamma \left(\frac{1}{2} \Delta_\pi + 1 \right) \right] m_{k_1 \dots k_N} = 0, \quad (45)$$

with

$$\Delta_\alpha m_{k_1 \dots k_N} = 2 \sum_{i_1, \dots, i_N=0}^3 \alpha(m_{i_1, \dots, i_N}) \prod_{l=1}^N \left[2 \left(\delta_{i_l k_l} + \delta_{i_l 0} + \delta_{k_l 0} - 2 \delta_{i_l 0} \delta_{k_l 0} - \frac{1}{2} \right) - 1 \right] m_{k_1 \dots k_N} \quad (46)$$

for $\alpha = \gamma, \pi, \dot{\pi}$. The conditions for the positivity of $-\Delta_\gamma$ agree with the BLP conditions for Λ_t in (43).

6 Conclusions

The following paper describes the decoherence of a qunit within the geometrical formulation of quantum mechanics. We show that the Markovian master equation for density operators is equivalent to the Fokker-Planck equation for quantum probability distributions.

It turns out that there are three natural generalizations of the qubit case, where we choose different operators in the place of the noise operators. For the Weyl operators, a certain choice of the decoherence rates results in the generator being a linear combination of squared modules of the Hamiltonian vector fields corresponding to the Weyl operators. For the Gell-Mann matrices, on the other hand, the generator of the evolution is the generalized Laplacian on $\mathbb{C}P^{n-1}$, similarly to the two-dimensional case [14]. It is easy to show that for the isotropic case this simplifies to the Laplace operator. Finally, in the case of tensor products of the Pauli matrices, we arrive at a simple equation describing the dependence between the decoherence rates and probability distribution present in the Kraus decomposition.

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